INHOMOGENOUS MULTISPECIES TASEP ON A RING WITH SPECTRAL PARAMETERS

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Abstract. We study an inhomogenous multispecies version of the Totally Asymmetric Simple Exclusion Process (TASEP) on a periodic oriented one dimensional lattice, which depends on two sets of parameters $(\boldsymbol{\tau}, \boldsymbol{\nu})$, attached to the particles. After discussing the Yang-Baxter integrability of our model, we study its (unnormalized) stationary measure. Motivated by the integrability of the model we introduce a further set of spectral parameters $\mathbf{z}$, attached to the sites of the lattice, and we uncover a remarkable underlying algebraic structure. We provide exact formulas for the stationary measure and prove the factorization of the stationary probability of certain configurations in terms of double Schubert polynomials in $(\boldsymbol{\tau}, \boldsymbol{\nu})$.

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1. Introduction

The Asymmetric Simple Exclusion process (ASEP) is a stochastic process that in the course of the last thirty years has gained the status of a paradigmatic model in the theory of far from equilibrium low dimensional systems [13]. The model describes the stochastic evolution of particles that occupy the sites of a one dimensional lattice under the exclusion condition, which means that each site can contain one particle at most. The dynamics involves jumps of the particles on neighboring sites with asymmetric rates for left or right jumping, modeling in this way the presence of an external driving force.

On one side the ASEP displays a rich phenomenology and has found a wide range of applications, going from the study of traffic flow, to that of surface growth, or sequence alignment (see [9] for a recent review of several of these applications). On the other side the ASEP is amenable to a variety of mathematical approaches, in part leading to complementary sets of results. Among these we mention Bethe Ansatz [20], quadratic algebras [16], combinatorics [10], orthogonal polynomials [33], random matrices [21], stochastic differential equations [11] and hydrodynamic limits [32].

In the present and in the following companion paper [7] we study a multispecies generalization of ASEP on a ring, i.e. on a periodic oriented one dimensional lattice. In this model, particles belong to different species (labeled by integers) and the exclusion condition is implemented by requiring each lattice site to be occupied by exactly one particle (at wish one can interpret particles of a given species as empty sites). The time evolution consists of swaps of neighboring particles: a particle of species α on the left swaps its position with a particle of species β on the right with transition rates $r_{\alpha,\beta}$ given by

$$r_{\alpha,\beta} = \begin{cases} \tau_\alpha - \nu_\beta & \alpha < \beta \\ 0 & \alpha \geq \beta \end{cases}$$

for some family of parameters $\boldsymbol{\tau} = \{\tau_\alpha\}_{\alpha\in\mathbb{Z}}, \boldsymbol{\nu} = \{\nu_\alpha\}_{\alpha\in\mathbb{Z}}$. In particular, since particles of higher species cannot overcome particles of lower species, we speak of a multispecies Totally Asymmetric Exclusion process (M-TASEP). As will be explained in Section 3, such a choice ensure the Yang-Baxter integrability of the M-TASEP. Our main focus in the present paper will be on the *stationary probability*, and moreover we will restrict to a system with a single particle per species on a ring of length N. On one hand this allows to use a light notation, on the other hand, as it will explained in [7] the results for more general species content can be derived from the ones presented here.

For some choices of the parameters $\boldsymbol{\tau}, \boldsymbol{\nu}$ the model has already appeared in the literature. The case $\nu_\alpha = \tau_\alpha$ has been considered by Karimipour in [22], where using a matrix product representation he showed that the stationary measure is uniform. Another case has appeared in the work of Rákos and Schütz [31]. They considered a system of N species of particles, each species moving to the right on empty sites with rates v_α, but exchange of particles is forbidden and since the particles cannot exchange position, one can assume each particle to be of a different species. In order to fit this model in our framework we identify particles of species $N+1$ as empty sites and to forbid the exchange of particles of successive species α, $\alpha+1$, which means $\nu_{N+1} = 0$, and $\nu_\alpha = \tau_{\alpha-1}$ for $\alpha \leq N$.

The main motivation of the present work come though from yet another particular case considered by Lam and Williams [25] in which all the parameters ν_α vanish. Lam and Williams conjectured that the stationary probabilities of the particles configurations, apart for a normalization factor called *partition function*, turn out to be polynomials in the parameters $\boldsymbol{\tau} = \{\tau_\alpha\}$, with positive integer coefficients. Actually they made an even stronger and more intriguing conjecture, namely that the unnormalized probability $\psi_{\mathbf{w}}(\boldsymbol{\tau})$ of any particle configuration $\mathbf{w}$ is a non negative integral sum of *Schubert polynomials* in the variables $\boldsymbol{\tau}$. On top of this they gave explicit formulas for certain components as products of Schubert polynomials [25, Conjecture 3 and 4](see Appendix A for the definition of Schubert polynomials). The weaker result on integrality and positivity of the coefficients of $\psi_{\mathbf{w}}(\boldsymbol{\tau})$ was soon settled in two steps. As a first step Ayyer and Linusson [6] gave a conjectural combinatorial expression of the integers coefficients as enumerating certain multiline queues previously introduced by Ferrari and Martin [17, 18]. Shortly later Arita and Mallick [4] proved Ayyer-Linusson conjecture by constructing a matrix product ansatz representation of $\psi_{\mathbf{w}}(\boldsymbol{\tau})$ which turns out to be equivalent to the multiline queues. Since then, many known results about stationary measure of the M-TASEP have been obtained by using the multiline queues (see for example [1, 2, 5]). Still the approach through multiline queues has given no insight for explaining the appearance of Schubert polynomials in this problem. Moreover the matrix representation of multispecies TASEP has been rederived recently in the framework of the Zamolodchikov tetrahedron equation [23, 24].

For generic $\boldsymbol{\nu}$, some of Lam and Williams conjectures extend in a natural way. It is convenient to think at the unnormalized probabilities $\psi_{\mathbf{w}}(\boldsymbol{\tau}, \boldsymbol{\nu})$ as components of a vector $\Psi_N(\boldsymbol{\tau}, \boldsymbol{\nu})$ in a basis labeled

by particle configurations $\mathbf{w}$, therefore in this paper we speak of "components" instead of "unnormalized probabilities". Let's specify the normalization of $\Psi_N(\boldsymbol{\tau},\boldsymbol{\nu})$ by fixing the component associated to the configuration $12\dots N$ as

$$\psi_{12\dots N}(\boldsymbol{\tau},\boldsymbol{\nu}) = \phi_N(\boldsymbol{\tau},\boldsymbol{\nu}), \tag{1}$$

with

$$\phi_N(\boldsymbol{\tau},\boldsymbol{\nu}) := \prod_{1\le\alpha<\beta\le N} (\tau_\alpha - \nu_\beta)^{\beta-\alpha-1}. \tag{2}$$

We have the following Theorem that shall be proven in the paper (it will be a corollary of Theorem 18)

***Theorem* 1.** *With the normalization given by eq.(1), the components $\psi_{\mathbf{w}}(\boldsymbol{\tau},\boldsymbol{\nu})$ are relative prime polynomials in $\boldsymbol{\tau},\boldsymbol{\nu}$, with integer coefficients.*

Moreover numerical computations at small sizes suggest the following

***Conjecture* 2.** *With the normalization given by eq.(1), the components $\psi_{\mathbf{w}}(\boldsymbol{\tau},-\boldsymbol{\nu})$ are polynomials in $\boldsymbol{\tau},\boldsymbol{\nu}$, with positive integer coefficients.*

A natural question to ask is whether these coefficients have any combinatorial origin. This amounts to ask whether there exist combinatorial objects, possibly generalizing the multiline queues of Ferrari and Martins, whose appropriately weighted enumerations coincide with $\psi_{\mathbf{w}}(\boldsymbol{\tau},-\boldsymbol{\nu})$. A related question is to construct a Matrix Product Ansatz [14] representation of the stationary measure.

In this paper we study the stationary measure through an approach which is based on ideas introduced by Di Francesco and Zinn-Justin in the context of the stochastic dense O(1) loop model [15]. For a system on a ring of length L we deform the master equation for the stationary measure by introducing scattering matrices that depends on L spectral parameters $\mathbf{z} = \{z_1,\dots,z_L\}$. The scattering matrices have a common stationary state $\Psi_N(\mathbf{z})$ that reduces to $\Psi_N(\boldsymbol{\tau},\boldsymbol{\nu})$ for $z_i = \infty$. We show that $\Psi_N(\mathbf{z})$ is solution of a set of exchange equations. Such equations involve certain divided difference operators $\pi_i(\alpha,\beta)$ (see eq.(28) for their definition) that generalize the isobaric divided difference operators of Lascoux and Schützenberger and whose commutation relation generalize to one satisfied by the generators of the $0-$Hecke algebra [27, 19].

By analyzing the exchange equations we prove exact expressions for the components associated to several configurations. Notably we show

that the component $\psi_{N(N-1)\dots 1}(\mathbf{z})$ associated to the configuration

$$N(N-1)\dots 21$$

factorizes in terms of polynomials $\mathfrak{S}^{r,s}(\mathbf{z})$ that correspond to a $\mathbf{z}$ deformations of certain Double Schubert polynomials, thus proving a generalization of one of Lam and Williams conjectures. As a byproduct of the analysis we show that many components factorize in terms of $\mathfrak{S}^{r,s}(\mathbf{z})$. Moreover, the knowledge of the explicit form of $\psi_{N(N-1)\dots 1}(\mathbf{z})$ allows to compute the so called partition function $\mathcal{Z}_N(\mathbf{z},\boldsymbol{\tau},\boldsymbol{\nu})$, i.e. the sum of all the components and also to show a remarkable factorization of the stationary measure that has already been proven for $\nu_\alpha = 0$ by Aas and Sjöstrand [2] using multiline queues enumerations.

The paper is organized as follows. In Section 2 we discuss briefly multispecies exclusion processes fixing the notations we use in the rest of the paper. In Section 3 we discuss the Yang-Baxter integrability of the multispecies exclusion processes, in particular we show how integrability leads to the exchange rates discussed above.

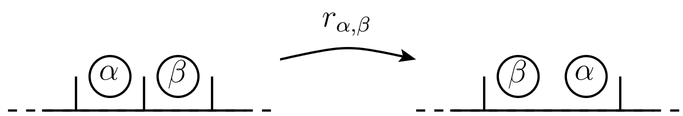
Sections 4 and 5 form the core of the paper. In Section 4 we deform the master equation for the stationary measure by introducing scattering operators that involve the spectral parameters. Then unique stationary measure of the scattering operators depends on the spectral parameters and reduces the the stationary measure of the original M-TASEP when the spectral parameters are set to 0. In the same section we show that the stationary measure of the scattering operators can be normalized in such a way to satisfy certain exchange equations. In Section 5 we start by analyzing the exchange equations, by expressing them in terms of divided difference operators acting on the configurations probabilities. Then in Section 5.2 we derive trivial factors of the components, this allows to compute $\psi_{12\dots N}(\mathbf{z})$ and to determine the degree $\Psi_N(\mathbf{z})$ as a polynomial in $\mathbf{z}$. In Section 5.3 we derive recursions relating $\Psi_N(\mathbf{z})$ and $\Psi_{N-1}(\mathbf{z})$. These will be used in Section 5.4 to provide an the formula for $\psi_{N(N-1)\dots 1}(\mathbf{z})$. Then in Section 5.5 we show the factorization property of the stationary measure and compute the partition function.

In Appendix A we present the definition of Double Schubert polynomials, while in Appendix B we gather some technical results used in the paper.

2. Multispecies exclusion processes

In *multispecies exclusion processes* each site of a periodic oriented lattice (a ring) is occupied by a particle, and the particles belong to

different species labeled by integers. The dynamics takes place in continuous time and consists of local updates of pairs of neighboring sites: if the site i is occupied by a particle of specie α and site $i+1$ is occupied by a particle of specie β then the exchange rate is $r_{\alpha,\beta}$.



Particles belonging to the same specie are considered as indistinguishable, therefore there is no rate $r_{\alpha,\alpha}$.

For a periodic lattice of length L, a configuration $\mathbf{w} = \{w_1 w_2 \dots w_L\}$ is specified by assigning to each site i an integer w_i corresponding to the specie of the particle occupying that site. Since the dynamics preserves the number of particles, the system is completely specified when we fix m_α, the number of particles of species α present on the lattice. If there are N different species of particles on the lattice, we can assume (up to relabeling of the species index) that $m_\alpha > 0$ for $1 \le \alpha \le N$ and call $\mathbf{m} = \{m_1, \dots, m_N\}$ the *species content*.

Example. Here is an example of a configuration for a system of size $L = 8$

$$\mathbf{w} = \{1,2,2,4,1,5,3,5\}, \ \mathbf{m}(\mathbf{w}) = \{2,2,1,1,2\}.$$

We call $\mathcal{H}_{\mathbf{m}}$ the vector space of states of content $\mathbf{m}$, with a preferred basis $v_{\mathbf{w}}$ labeled by configurations $\mathbf{w}$ such that $\mathbf{m}(\mathbf{w}) = \mathbf{m}$. In order to write explicitly the Markov matrix defining the stochastic evolution of the system on $\mathcal{H}_{\mathbf{m}}$, it is convenient to introduce $\mathcal{H}^L$, the space with unconstrained content, which has a preferred basis labeled by all the configurations $\mathbf{w}$ of length $L(w) = L$. Such space $\mathcal{H}^L$ has a natural tensor product structure

$$\mathcal{H}^L = V_1 \otimes \cdots \otimes V_i \otimes \cdots \otimes V_L$$

where $V_i \simeq V \simeq \mathbb{C}^\infty$, with a preferred basis $\{v_\alpha | \alpha \in \mathbb{N}\}$. The state space $\mathcal{H}_{\mathbf{m}}$ of a M-TASEP of content $\mathbf{m}$ is naturally embedded as a subspace in $\mathcal{H}^L$ with $L = |\mathbf{m}| := \sum_{\alpha=1}^N m_\alpha$.

Call $p_{\mathbf{w}}(t)$ the probability of having a the configuration $\mathbf{w}$ of content $\mathbf{m}$ at time t. The time evolution of $p_{\mathbf{w}}(t)$ is determined by the Master equation

$$\frac{d}{dt}\mathcal{P}_{\mathbf{m}}(t) = \mathcal{M}\mathcal{P}_{\mathbf{m}}(t)$$

where the probabilities $p_{\mathbf{w}}(t)$ are gathered in the vector

$$\mathcal{P}_{\mathbf{m}}(t) := \sum_{\mathbf{w}|\mathbf{m}(\mathbf{w})=\mathbf{m}} p_{\mathbf{w}}(t) v_{\mathbf{w}}$$

and the Markov matrix $\mathcal{M}$ is written as a sum of local exchange terms

$$\mathcal{M} = \sum_{i=1}^{L} h_i. \tag{3}$$

The operator $h \in \mathrm{End}(V \otimes V)$, which accounts for the local exchange rates on two consecutive sites, reads

$$h = \sum_{\alpha,\beta\in\mathbb{Z}} r_{\alpha,\beta} T^{(\alpha,\beta)} \tag{4}$$

with

$$T^{(\alpha,\beta)} = E^{(\beta,\alpha)} \otimes E^{(\alpha,\beta)} - E^{(\alpha,\alpha)} \otimes E^{(\beta,\beta)}, \tag{5}$$

and the elementary operators $E^{(\alpha,\beta)} \in \mathrm{End}(V)$ act on the basis $\{v_\gamma\}$ of V by $E^{(\alpha,\beta)} v_\gamma = \delta_{\beta,\gamma} v_\alpha$. The operators h_i act locally on the tensor product $V_i \otimes V_{i+1}$

$$h_i = \mathbf{1}_1 \otimes \cdots \mathbf{1}_{i-1} \otimes h \otimes \mathbf{1}_{i+2} \otimes \cdots \otimes \mathbf{1}_L.$$

In the present paper we will be concerned only with the stationary measure $\mathcal{P}_{\mathbf{m}}$. For sufficiently generic rates $r_{\alpha,\beta}$ such measure is unique and is given by the solution of the master equation

$$\mathcal{M}\mathcal{P}_{\mathbf{m}} = 0. \tag{6}$$

The approach we will adopt consists in deforming the previous master equation. In order to this we have first to discuss the integrability of the M-TASEP.

3. Integrability

The standard way to show the Yang-Baxter integrability of an operator like $\mathcal{M}$, given by the sum of local operators, is to find $\check{R}_i(x,y)$ matrices, acting on $V_i \otimes V_{i+1}$, such that

$$\begin{aligned} &\check{R}_i(x,x) = \mathbf{1} \\ &\check{R}_i(x,y)\check{R}_i(y,x) = \mathbf{1} \\ &\frac{d}{dx}\check{R}_i(x,y)|_{x=y=c} \propto h_i \end{aligned} \tag{7}$$

and such that they satisfy the braided Yang-Baxter equation

$$\check{R}_i(y,z)\check{R}_{i+1}(x,z)\check{R}_i(x,y) = \check{R}_{i+1}(x,y)\check{R}_i(x,z)\check{R}_{i+1}(y,z). \tag{8}$$

Motivated by the fact that h is itself the sum of more elementary operators $T^{(\alpha,\beta)}$, we search $\check{R}$ matrices of the baxterized form

$$\check{R}_i(x,y) = \mathbf{1} + \sum_{\alpha,\beta\in\mathbb{Z}} g_{\alpha,\beta}(x,y) T_i^{(\alpha,\beta)}. \tag{9}$$

The following result is probably well-known, but apparently never stated explicitly

***Theorem* 3.** *Assume that $\forall\alpha,\beta$ $g_{\alpha,\beta}(x,y)$ do not vanish identically. Then, up to reparametrization of the spectral variables (x,y) and re-ordering of the species labels, the baxterized solutions of eqs.(7,8) are labeled by a parameter q and read*

$$g_{\alpha,\beta}(x,y)=\frac{x-y}{1-(q+q^{-1})y+xy}q^{sign(\alpha-\beta)}. \tag{10}$$

Remark 1. The solution corresponding to (10) is nothing else than the baxterization of the Hecke algebra, indeed the operators

$$E_i=\sum_{\alpha,\beta\in\mathbb{Z}} q^{\mathrm{sign}(\alpha-\beta)}T_i^{(\alpha,\beta)} \tag{11}$$

satisfy the Hecke relations [3]

$$\begin{aligned} E_i^2 &= -(q+q^{-1})E_i \\ [E_i,E_j] &= 0 \text{ for } |i-j|>1 \\ E_iE_{i+1}E_i-E_i &= E_{i+1}E_iE_{i+1}-E_{i+1}. \end{aligned} \tag{12}$$

In order to obtain a richer family of solutions we set to zero the function $g_{\alpha,\beta}(x,y)$ for $\alpha>\beta$.

***Theorem* 4.** *Suppose that $g_{\alpha,\beta}(x,y)=0$ for $\alpha>\beta$, while $g_{\alpha,\beta}(x,y)$ not identically zero for $\alpha<\beta$. Then the most general solution of eqs.(7,8), of the form (9), is given for $\alpha<\beta$ by*

$$g_{\alpha,\beta}(x,y)=g(x,y|\tau_\alpha,\nu_\beta):=1-\frac{f(x|\tau_\alpha,\nu_\beta)}{f(y|\tau_\alpha,\nu_\beta)}, \tag{13}$$

with

$$f(x|\tau,\nu)=\frac{x-\tau}{x-\nu}.$$

Proof. Call YB the difference between left and right hand side of the Yang-Baxter equations (8), then we look for the solutions of the equations $YB_{\Theta}^{\Theta'}=0$ with the multi-indices of $\Theta=\{\theta_1,\theta_2,\theta_3\}$ and $\Theta'=\{\theta'_1,\theta'_2,\theta'_3\}$ which are related by a permutation.

If we restrict the elements of Θ to the set $\{\alpha,\beta\}$, then what we obtain are the Yang-Baxter equations of a problem with just the two species α and β. Direct inspection of these equations shows that their solution take the form $g_{\alpha,\beta}(x,y)=1-\frac{f_{\alpha,\beta}(x)}{f_{\alpha,\beta}(y)}$, where at this point $f_{\alpha,\beta}(x)$ is an arbitrary function. In order to see how the different functions $f_{\alpha,\beta}(x)$ are related one has to look at equation in which all the three species

α, β and γ appear. Let's look at the equations $YB^{\{\beta,\gamma,\alpha\}}_{\{\alpha,\beta,\gamma\}} = 0$ and $YB^{\{\gamma,\alpha,\beta\}}_{\{\alpha,\beta,\gamma\}} = 0$, which read respectively

$$(g_{\alpha,\beta}(y,z)g_{\alpha,\gamma}(x,z) - g_{\alpha,\beta}(x,z)g_{\alpha,\gamma}(y,z))(g_{\beta,\gamma}(x,y) - 1) = 0, \tag{14}$$

$$(g_{\alpha,\gamma}(x,z)g_{\beta,\gamma}(x,y) - g_{\alpha,\gamma}(x,y)g_{\beta,\gamma}(x,z))(g_{\alpha,\beta}(y,z) - 1) = 0. \tag{15}$$

If $\alpha < \beta, \gamma$ then the functions $g_{\alpha,\beta}(x,y)$ and $g_{\alpha,\gamma}(x,y)$ are not identically zero and one can rewrite eq.(14) as

$$\frac{g_{\alpha,\beta}(x,z)}{g_{\alpha,\gamma}(x,z)} = \frac{g_{\alpha,\beta}(y,z)}{g_{\alpha,\gamma}(y,z)} = F_1^{\alpha,\beta,\gamma}(z) \tag{16}$$

for some $F_1^{\alpha,\beta,\gamma}(z)$ which depends only on z. Analougously if $\gamma > \alpha, \beta$, eq.(15) can be rewritten as

$$\frac{g_{\alpha,\gamma}(x,y)}{g_{\beta,\gamma}(x,y)} = \frac{g_{\alpha,\gamma}(x,z)}{g_{\beta,\gamma}(x,z)} = F_2^{\alpha,\beta,\gamma}(x) \tag{17}$$

for some $F_2^{\alpha,\beta,\gamma}(x)$ which depends only on x. Once expressed in terms of $f_{\alpha,\beta}(x), f_{\alpha,\gamma}(x)$ and $f_{\beta,\gamma}(x)$, eqs.(16,17) imply that these functions are related one to the other by projective transformations. Therefore without loosing generality we can assume

$$f_{\alpha,\beta}(x) = \frac{x - \tau_{\alpha,\beta}}{x - \nu_{\alpha,\beta}}$$

When this form of $f_{\alpha,\beta}(x), f_{\alpha,\gamma}(x)$ and $f_{\beta,\gamma}(x)$ is plugged into eq.(16) one finds that $\tau_{\alpha,\beta} = \tau_{\alpha,\gamma}$, while from eq.(17) one finds $\nu_{\alpha,\gamma} = \nu_{\beta,\gamma}$ which mean that the parameters τ and ν depend only on the first and on the second index respectively. In order to conclude the proof it is sufficient to check that with the choice $g_{\alpha,\beta}(x,y)$ of eq.(13) all the other components of the Yang-Baxter equations automatically vanish, which is the case. $\square$

Remark 2. Actually with just a little more annoying work one can relax the hypothesis of Theorem 4. It is enough to assume that for some $\gamma < \delta$, $g_{\delta,\gamma}(x,y) = 0$, while $g_{\gamma,\delta}(x,y)$ not identically zero. Then, up to relabeling of the species, one can deduce that $g_{\alpha,\beta}(x,y) = 0$ for $\alpha > \beta$.

Remark 3. The solution of the Yang-Baxter equation of Theorem 4 was first found in an implicit form in [8, Appendix B]. Recently such solution has found a nice algebraic formulation and generalization in [12].

The derivative of $\check{R}_i(x,y)$ specialized in $x=y=c$ reads

$$h_i(c) = c^2 \frac{d}{dx}\check{R}_i(x,y)|_{x=y=c} = \sum_{\gamma\le\alpha<\beta\le\delta} \left(\frac{c^2}{\tau_\alpha - c} - \frac{c^2}{\nu_\beta - c}\right) T_i^{(\alpha,\beta)}. \tag{18}$$

By setting $c=\infty$ we obtain the exchange rates

$$r_{\alpha\ge\beta} = 0, \qquad r_{\alpha<\beta} = \tau_\alpha - \nu_\beta. \tag{19}$$

This is the class of models whose stationary measure we analyze in the rest of the paper. Certain particular cases of this class have already appeared in the literature.

In the introduction we have already mentioned the work of Lam and Williams [25] about the case $\nu_\alpha = 0$, which has been one of the main motivation of the present paper.

In [22] Karimpour studied the case in which N species of particles moves on a ring with empty spaces with the following rules: a particle of type α moves to an ampty site with rate v_α (which is interpreted as the "speed" of this specie), while two particles of species α and β exchange position with rate $v_\alpha - v_\beta$ if it is positive or they do not move (without loosing generality one can assume $v_\alpha \ge v_{\alpha+1}$. This model would correspond in our language to a system with $N+1$ species, the $N+1$st corresponding to empty sites and parameters $\nu_\alpha = \tau_\alpha = v_\alpha$ for $1 \le \alpha \le N$ and $\nu_{N+1} = 0$. Using a Matrix Product Ansatz, Karimpour showed that the stationary probability is simply the uniform measure, this result can also be easily recovered with our approach that will be explained in Section 5.1.

Another case has appeared in the work of Rákos and Schütz [31]. They considered a system of N species of particles, each species moving to the right on empty sites with rates v_α as in Karimpour's models, but exchange of particles is forbidden and since the particles cannot exchange position, one can assume each particle to be of a different species. In order to fit this model in our language it is enough to identify particles of species $N+1$ as empty sites and to forbid the exchange of particles of successive species α, $\alpha+1$, which means $\nu_{N+1} = 0$, and $\nu_\alpha = \tau_{\alpha-1}$ for $\alpha \le N$. In this way, the subset of configurations in which (up to cyclic permutations) particles with labels less then $N+1$ are ordered increasingly is absorbing and preserved by time evolution.

4. Exchange equations

The goal of this section is to use the integrability to deform the stationary eqs.(6) by introducing the spectral parameters. The starting

point are the scattering matrices $S_i(\mathbf{z})$, defined by

$$S_i(\mathbf{z}) := \mathcal{R}\check{R}_{i-2}(z_i, z_{i-1}) \dots \check{R}_{i+1}(z_i, z_{i+2})\check{R}_i(z_i, z_{i+1}) \tag{20}$$

where $\mathcal{R} \in \mathrm{End}(V_1 \otimes V_2 \otimes \cdots \otimes V_N)$ is the operator that "rotates" our system

$$\mathcal{R}(v_1 \otimes v_2 \otimes \cdots \otimes v_{N-1} \otimes v_N) := (v_N \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{N-1}). \tag{21}$$

Thanks to the Yang-Baxter equation (8) it is easy to verify that the scattering matrices commute among themselves

$$[S_i(\mathbf{z}), S_j(\mathbf{z})] = 0. \tag{22}$$

Moreover we have the following important

***Proposition* 5.** *The scattering matrices $S_i(\mathbf{z})$, acting on $\mathcal{H}_{\mathbf{m}}$, have a single common eigenvector $\Psi_{\mathbf{m}}(\mathbf{z})$ of eigenvalue 1 for any i.*

Proof. By choosing $\mathbf{z}$, $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ such that $\forall j \neq i$, $0 < g_{a,b}(z_i, z_j) < 1$ we have that the matrices $S_i(\mathbf{z})$ are irreducible stochastic matrices in any sector $\mathcal{H}_{\mathbf{m}}$, hence each of them has a single right eigenvector with eigenvalue 1 in $\mathcal{H}_{\mathbf{m}}$, that we call $\Psi_i(\mathbf{z})$. It remains to show that $\Psi_i(\mathbf{z}) = \Psi_j(\mathbf{z})$ for $i \neq j$. Suppose by absurd that this is not the case. Since the matrices $S_i(\mathbf{z})$ commute, all the vectors $\Psi_j(\mathbf{z})$ are right eigenvectors of any $S_i(\mathbf{z})$. By absurd therefore one should have that for some $i \neq j$, $S_i(\mathbf{z})\Psi_j(\mathbf{z}) = \lambda_{i,j}\Psi_j(\mathbf{z})$ with $\lambda_{i,j} \neq 1$. But this would mean that $\Psi_j(\mathbf{z})$ is orthogonal to the common left eigenvector (whose entries are all equal), which is impossible by the Perron-Frobenius theorem. $\square$

Since the entries of $S_i(\mathbf{z})$ are rational functions of the variables $\mathbf{z}$, $\Psi_{\mathbf{m}}(\mathbf{z})$ can be normalized in such a way that its entries are polynomials of such variables. The key result that allows to compute $\Psi_{\mathbf{m}}(\mathbf{z})$ is the following

***Theorem* 6** (Exchange equations)**.** *Let $\Psi_{\mathbf{m}}(\mathbf{z})$ be the unique (up to scalar multiplication) common eigenvector of $S_i(\mathbf{z})$ in the sector $\mathcal{H}_{\mathbf{m}}$, with eigenvalue 1 and normalized in such a way that its components are polynomials of $\mathbf{z}$ of minimal degree. Then $\Psi_{\mathbf{m}}(\mathbf{z})$ satisfies the following exchange equations*

$$\boxed{\check{R}_i(z_i, z_{i+1})\Psi_{\mathbf{m}}(\mathbf{z}) = s_i \circ \Psi_{\mathbf{m}}(\mathbf{z})} \tag{23}$$

where s_i exchange the variables $z_i \leftrightarrow z_{i+1}$.

Proof. Take $j \neq i, i+1$, then from the Yang-Baxter equation (8) we immediately find that

$$R_i(z_i, z_{i+1})S_j(\mathbf{z}) = (s_i \circ S_j(\mathbf{z}))R_i(z_i, z_{i+1}).$$

This means that $R_i(z_i, z_{i+1})\Psi_{\mathbf{m}}(\mathbf{z})$ is an eigenvector of $(s_i \circ S_j(\mathbf{z}))$ with eigenvalue equal to 1, therefore by uniqueness it must be proportional to $s_i \circ \Psi_{\mathbf{m}}(\mathbf{z})$.

$$\check{R}_i(z_i, z_{i+1})\Psi_{\mathbf{m}}(\mathbf{z}) = c_i(\mathbf{z}) s_i \circ \Psi_{\mathbf{m}}(\mathbf{z}).$$

The proportionality factor $c_i(\mathbf{z})$ is a rational function of $\mathbf{z}$ and, since $\Psi_{\mathbf{m}}(\mathbf{z})$ is supposed to be of minimal degree, its denominator part can possibly come only from the poles of $\check{R}_i(z_i, z_{i+1})$, in particular it depends only on z_i, z_{i+1} in a factorized form, i.e. $c_i(\mathbf{z}) = \frac{\bar{c}_i(\mathbf{z})}{k_1^{(i)}(x_i)k_2^{(i)}(x_{i+1})}$, where $k_1^{(i)}(x_i)$ is some product of $(x_i - \tau_\alpha)$, while $k_2^{(i)}(x_{i+1})$ is some product of $(x_i - \nu_\beta)$. From $\check{R}_i(z_i, z_{i+1})\check{R}_i(z_{i+1}, z_i) = 1$ it follows that $c_i(\mathbf{z})\,(s_i \circ c_i(\mathbf{z})) = 1$ and from $\check{R}_i(z, z) = 1$ it follows $c_i(\mathbf{z})|\{z_i = z_{i+1}\} = 1$. Combining these information we conclude that $c_i(\mathbf{z})$ must be of the form $c_i(\mathbf{z}) = \frac{k_1^{(i)}(x_{i+1})k_2^{(i)}(x_i)}{k_1^{(i)}(x_i)k_2^{(i)}(x_{i+1})}$. Remark that for any pair (i, j), $k_1^{(i)}(x)$ and $k_2^{(j)}(x)$ have no common factors.

Now specialize $z_1 = z$ and for $i \neq 1$, $z_i = w$, by repeatedly applying the exchange equation it follows that

$$\check{R}_N(z, w) \dots \check{R}_1(z, w)\Psi_{\mathbf{m}}(z, w, \dots, w) = \prod_{i=1}^{N} \frac{k_1^{(i)}(w)k_2^{(i)}(z)}{k_1^{(i)}(z)k_2^{(i)}(w)} \Psi_{\mathbf{m}}(z, w, \dots, w)$$

Contracting both sides of the previous equation with the dual eigenvector we find that

$$\prod_{i=1}^{N} \frac{k_1^{(i)}(w)k_2^{(i)}(z)}{k_1^{(i)}(z)k_2^{(i)}(w)} = 1$$

which, in view of the absence of common factors between $k_1^{(i)}(x)$ and $k_2^{(j)}(x)$, forces all the $k_\alpha^{(i)}(x)$ to be equal to 1. $\square$

Corollary 7. *The exchange equations (23) have a unique solution up to multiplication by a symmetric function of* $\mathbf{z}$.

At this point we can easily relate $\Psi_{\mathbf{m}}(\mathbf{z})$ to $\mathcal{P}_{\mathbf{m}}$, the stationary measure in the sector $\mathcal{H}_{\mathbf{m}}$: for $\mathbf{z} = \infty$ the two are just proportional[1]

$$\Psi_{\mathbf{m}}(\infty) \propto \mathcal{P}_{\mathbf{m}}. \tag{24}$$

[1] If $P(\mathbf{x})$ is a polynomial of total degree D in the variables $\mathbf{x} = x_1, x_2, \dots, x_L$, by $P(\infty)$ we mean the coefficient of the monomial of top degree of $P(\mathbf{x})|_{x_1=x_2=\dots=x_L=x}$, i.e.

$$P(\infty) := \lim_{x\to\infty} x^{-D} P(\mathbf{x})|_{x_1=x_2=\cdots,x_L=x}.$$

Indeed by differentiating eq.(23) with respect to z_i and then setting $\mathbf{z} = c$ we get

$$h_i(c)\Psi_{\mathbf{m}}(c) + c^2\partial_i\Psi_{\mathbf{m}}(c) = c^2\partial_{i+1}\Psi_{\mathbf{m}}(c)$$

from which we see that the sum of the terms $h_i(c)\Psi_{\mathbf{m}}(c)$ is telescopic and at the end we obtain

$$\sum_{i=1}^{L} h_i(c)\Psi_{\mathbf{m}}(c) = 0. \tag{25}$$

In the rest of the paper we concentrate on $\Psi_N(\mathbf{z})$ the solution of the exchange eq.(23) in the case $m_i = 1$ for $1 \leq i \leq N$ and zero otherwise. We will be able to derive several properties of $\Psi_N(\mathbf{z})$, and through a straightforward specialization $\mathbf{z} = \infty$ we will settle some of the questions about Ψ_N mentioned in the Introduction.

5. Solution of the exchange equations

5.1. **The exchange equations in components.** As a first step we expand the exchange equations (23) into the basis $v_{\mathbf{w}}$

$$\Psi_{\mathbf{m}}(\mathbf{z}) = \sum_{\ell(\mathbf{w})=N} \psi_{\mathbf{w}}(\mathbf{z}) v_{\mathbf{w}}. \tag{26}$$

The components $\psi_{\mathbf{w}}(\mathbf{z})$ correspond to a deformation of the (unnormalized) stationary probabilities of the configurations $\mathbf{w}$. In order to write the exchange equation in a compact way it is convenient to introduce the natural action of the symmetric group $\mathcal{S}_N$ on particles configurations, for $\sigma \in \mathcal{S}_N$

$$\sigma\{w_1, \ldots, w_N\} = \{w_{\sigma(1)}, \ldots, w_{\sigma(N)}\}.$$

Then the exchange equations between positions $(i, i+1)$ become

$$\boxed{\psi_{\mathbf{w}}(\mathbf{z}) = \pi_i(w_{i+1}, w_i)\psi_{s_i\circ\mathbf{w}}(\mathbf{z}) \quad \text{if} \quad w_i < w_{i+1}.} \tag{27}$$

where $\pi_i(\beta, \alpha)$ are isobaric divided difference operators in the variables $f(z|\tau_\alpha, \nu_\beta)$ defined by

$$\boxed{\begin{aligned} \pi_i(\beta,\alpha)G(\mathbf{z}) &= f(z_{i+1}|\tau_\alpha,\nu_\beta)\frac{G(\mathbf{z}) - s_i\circ G(\mathbf{z})}{f(z_i|\tau_\alpha,\nu_\beta) - f(z_{i+1}|\tau_\alpha,\nu_\beta)} \\ &= \frac{(z_{i+1}-\tau_\alpha)(z_i-\nu_\beta)}{\tau_\alpha - \nu_\beta}\frac{G(\mathbf{z}) - s_i\circ G(\mathbf{z})}{z_i - z_{i+1}} \end{aligned}} \tag{28}$$

It is not difficult to realize that the system of equations (27) is cyclic: suppose a component associated to a configurations $\mathbf{w}$ is known, all

the other components can be obtained by the action of the operators $\pi_i(\alpha, \beta)$.

Before proceeding further in the analysis of eqs.(27) we want to spend a couple of words about the operators $\pi_i(\beta, \alpha)$. These operators satisfy relations that generalize the ones satisfied by the generators of the $0-$Hecke algebra

$$\begin{aligned} \pi_i(\alpha,\beta)\pi_j(\gamma,\delta) &= \pi_j(\gamma,\delta)\pi_i(\alpha,\beta) \qquad |i-j|>1 \\ \pi_i(\alpha,\beta)\pi_i(\gamma,\delta) &= -\pi_i(\alpha,\beta) \\ \pi_i(\beta,\gamma)\pi_{i+1}(\alpha,\gamma)\pi_i(\alpha,\beta) &= \pi_{i+1}(\alpha,\beta)\pi_i(\alpha,\gamma)\pi_{i+1}(\beta,\gamma). \end{aligned} \tag{29}$$

In particular we remark that the last equation is a braided Yang-Baxter equation. The $0-$Hecke algebra is recovered in the case we choose $\tau_\alpha = \tau$ and $\nu_\alpha = \nu$ $\forall\alpha$, in which case the operators $\pi_i(\alpha, \beta)$ become independent of the labels α, β and correspond to the more common isobaric divided difference operators π_i in the variable $\frac{1-\tau x}{1+\nu x}$ [26]. On the other hand, many of the remarkable properties of the operators π_i get generalized to the operators $\pi_i(\alpha, \beta)$.

5.2. **Trivial factors.** From eq.(27) we already know that as a polynomial in the spectral parameters, the component $\psi_{\mathbf{w}}(\mathbf{z})$ has factors $(z_{i+1} - \tau_{w_i})(z_i - \nu_{w_{i+1}})$ whenever $w_i < w_{i+1}$

$$\psi_{\mathbf{w}}(\mathbf{z}) = (z_{i+1} - \tau_{w_i})(z_i - \nu_{w_{i+1}})\tilde{\psi}_{\mathbf{w}}(\mathbf{z}) \tag{30}$$

and the factor $\tilde{\psi}_{\mathbf{w}}(\mathbf{z})$ is a polynomial symmetric under exchange $z_i \leftrightarrow z_{i+1}$. We want to use this remark in order to determine for each component $\psi_{\mathbf{w}}(\mathbf{z})$ as many "trivial" factors of the form $(z_i - \tau_\alpha)$ or $(z_i - \nu_\beta)$ as possible.

***Proposition* 8.** *1) Suppose that given $j < k$, the configuration $\mathbf{w}$ is such that $w_i < w_k$ for $j \le i < k$, then $\psi_{\mathbf{w}}(\mathbf{z})$ is divisible by $(z_j - \nu_{w_k})$.*
2) Suppose that given $j < k$, the configuration $\mathbf{w}$ is such that $w_i > w_j$ for $j < i \le k$, then $\psi_{\mathbf{w}}(\mathbf{z})$ is divisible by $(z_k - \tau_{w_j})$.

Proof. We just prove point 1), point 2) being completely analogous. We proceed by a simple induction on the difference $h = k - j$. For $h = 1$ the statement follows immediately from eq.(27). Now suppose the statement true for h and take $k - j = h + 1$. Since $w_k > w_{k-1}$ we know that $\psi_{\mathbf{w}}(\mathbf{z}) = \pi_{k-1}(w_k, w_{k-1})\psi_{s_{k-1}\circ\mathbf{w}}(\mathbf{z})$. By induction we know that $\psi_{s_{k-1}\circ\mathbf{w}}(\mathbf{z})$ is divisible by $(z_j - \nu_{w_k})$; on the other hand, since $\pi_{k-1}(w_k, w_{k-1})$ acts only on the variables z_{k-1}, z_k we conclude that $(z_j - \nu_{w_k})$ divides also $\psi_{\mathbf{w}}(\mathbf{z})$. □

An immediate corollary of the previous Proposition is the following

***Corollary* 9.** *1) If $w_i \neq N$ then $\psi_{\mathbf{w}}(\mathbf{z})$ is divisible by $(z_i - \nu_N)$.*
2) If $w_i \neq 1$ then $\psi_{\mathbf{w}}(\mathbf{z})$ is divisible by $(z_i - \tau_1)$.

A particularly favorable situation is when the configuration $\mathbf{w}$ presents a subset of consecutive increasing entries. For a finite set of integers I, a configuration $\mathbf{w}$ and $1 \leq j, k \leq N$ define

$$G_{I;\mathbf{w};j,k}(\mathbf{z}) = \prod_{i=j}^{k} \left(\prod_{\substack{\alpha \in I \\ w_i > \alpha}} (z_i - \tau_\alpha) \prod_{\substack{\alpha \in I \\ w_i < \alpha}} (z_i - \nu_\alpha) \right). \tag{31}$$

Then we have the following

***Proposition* 10.** *Suppose that for some $j < k$, the configuration $\mathbf{w}$ is such that $w_i < w_\ell$ for $j \leq i < \ell \leq k$, then the component $\psi_{\mathbf{w}}(\mathbf{z})$ is of the form*

$$\psi_{\mathbf{w}}(\mathbf{z}) = \tilde{\psi}_{\mathbf{w}}(\mathbf{z}) G_{\mathbf{w}_{[j,k]};\mathbf{w};j,k}(\mathbf{z}), \tag{32}$$

where $\tilde{\psi}_w(\mathbf{z})$ is a polynomial symmetric in the variables $z_j, z_{j+1}, \ldots, z_k$ and $\mathbf{w}_{[j,k]} = \{w_j, w_{j+1}, \ldots, w_k\}$.

Proof. The presence of the factor $G_{\mathbf{w}_{[j,k]};\mathbf{w};j,k}(\mathbf{z})$ is ensured by Proposition 8. Since $w_i < w_{i+1}$, $G_{\mathbf{w}_{[j,k]};\mathbf{w};j,k}(\mathbf{z})$ is given by $(z_{i+1} - \tau_{w_i})(z_i - \nu_{w_{i+1}})$ times a function symmetric under s_i. Therefore from eq.(30) we conclude that $\tilde{\psi}_w(\mathbf{z})$ is symmetric under s_i. $\square$

Now consider the configuration $12\ldots N$. For such configuration we have

$$\psi_{12\ldots N}(\mathbf{z}) = \tilde{\psi}(\mathbf{z}) \prod_{i=1}^{N} \left(\prod_{\alpha=1}^{i-1} (z_i - \tau_\alpha) \prod_{\alpha=i+1}^{N} (z_i - \nu_\alpha) \right). \tag{33}$$

where $\tilde{\psi}(\mathbf{z})$ is symmetric in all the spectral parameters. Any other component can be obtained from $\psi_{12\ldots N}(\mathbf{z})$ through the action of the operators $\pi_i(\alpha, \beta)$, which preserve symmetric factors. Therefore $\tilde{\psi}(\mathbf{z})$ appears as a factor of all the components. This means that in the minimal degree solution of the exchange equations, we can assume $\tilde{\psi}(\mathbf{z})$ to be a constant in the spectral parameter. We choose the normalization of $\Psi_N(\mathbf{z})$ by setting

$$\psi_{12\ldots N}(\mathbf{z}) = \phi_N(\boldsymbol{\tau}, \boldsymbol{\nu}) \prod_{i=1}^{N} \left(\prod_{\alpha=1}^{i-1} (z_i - \tau_\alpha) \prod_{\alpha=i+1}^{N} (z_i - \nu_\alpha) \right). \tag{34}$$

In particular in this way we have been able to fix the degree of $\Psi_N(\mathbf{z})$ as a polynomial in any of the spectral parameters z_i.

***Corollary* 11.** *The degree of $\Psi_{\mathbf{m}}(\mathbf{z})$ as a polynomial in z_i for any i is equal to $N-1$.*

Let us discuss briefly the particular case $\tau_\alpha = \nu_\alpha$. In such a case it is simple to see that

$$\pi_i(\alpha,\beta)(z_i-\nu_\alpha)^{-1}(z_{i+1}-\nu_\beta)^{-1} = (z_i-\nu_\beta)^{-1}(z_{i+1}-\nu_\alpha)^{-1}.$$

This implies that

$$\psi_{\mathbf{w}}(\mathbf{z}) = \prod_{1\le i\le L}(z_i-\nu_{w_i})^{-1} \tag{35}$$

is solution of the exchange eqs.(27)[2]. This is consistent with [22], indeed by setting $\mathbf{z}=\infty$ we simply obtain that the stationary measure is uniform.

5.3. **Recursions.** We have already seen in Corollary 9 that if we specialize $z_i=\tau_1$ or $z_i=\nu_N$, then all the components whose configuration doesn't present a particle of specie respectively 1 or N at position i are equal to zero. Here we want to characterize all the other components under the same specialization. The first step is to compare the specializations of $\Psi_N(\mathbf{z})$ at different positions. Let us define the operators $\hat{s}_i \in \mathrm{End}(\mathbb{C}[\mathbf{z}]\otimes\mathcal{H}^N)$

$$\hat{s}_i\left[f(\mathbf{z})v_{\mathbf{w}}\right] = s_i f(\mathbf{z})v_{s_i\mathbf{w}}$$

i.e. the map $\hat{s}_i$ transpose at the same time the variables z_i, z_{i+1} and the particles at position i and $i+1$. There is no reason for the operators $\hat{s}_i$ to preserve $\Psi_N(\mathbf{z})$ and indeed in general we have

$$\hat{s}_i\Psi_N(\mathbf{z}) \neq \Psi_N(\mathbf{z}),$$

but, upon specializations $z_i=\tau_1$ or $z_i=\nu_N$ we have the following

***Proposition* 12.**

$$\hat{s}_i\Psi_N(\mathbf{z})|_{z_i=\tau_1} = \Psi_N(\mathbf{z})|_{z_i=\tau_1} \tag{36}$$

$$\hat{s}_i\Psi_N(\mathbf{z})|_{z_i=\nu_N} = \Psi_N(\mathbf{z})|_{z_i=\nu_N} \tag{37}$$

Proof. Once written in components, eq.(36) just states that

$$\psi_{\mathbf{w}}(\dots,z_i,z_{i+1},\dots)|_{z_i=\tau_1} = \psi_{s_i\mathbf{w}}(\dots,z_{i+1},z_i,\dots)|_{z_i=\tau_1}. \tag{38}$$

[2]In order to make $\psi_{\mathbf{w}}(\mathbf{z})$ a polynomial is sufficient to multiply it by

$$\prod_{1\le i\le N}\prod_{1\le\alpha\le N}(z_i-\nu_\alpha).$$

The previous equation is obvious for $w_i \neq 1$, since both side vanish. It remains to show the case $w_i = 1$. From eq.(27) it follows that $\forall \mathbf{w}$, $\psi_{\mathbf{w}} + \psi_{s_i\mathbf{w}}$ is symmetric in $z_i \leftrightarrow z_{i+1}$

$$\psi_{\mathbf{w}}(z_i, z_{i+1}) + \psi_{s_i\mathbf{w}}(z_i, z_{i+1}) = \psi_{\mathbf{w}}(z_{i+1}, z_i) + \psi_{s_i\mathbf{w}}(z_{i+1}, z_i) \tag{39}$$

If $w_i = 1$ then $w_{i+1} \neq 1$ and setting $z_i = \tau_1$ we have that the terms $\psi_{s_i\mathbf{w}}(z_i, z_{i+1})$ and $\psi_{\mathbf{w}}(z_{i+1}, z_i)$ vanish and one remains with eq.(38). The proof of eq.(37) follows the same lines. □

Now let us define two insertion operators on configurations $\mathbf{w}$ of length $N-1$

- Υ_j^N applied to $\mathbf{w}$ inserts the entry N between w_{j-1} and w_j.
- Υ_j^1 applied to $\mathbf{w}$ inserts the entry 1 between w_{j-1} and w_j and increases all the other entries by 1.

For example

$$\Upsilon_3^6\, 21534 = 216534,\ \ \Upsilon_3^1\, 21534 = 321643.$$

and extend it to a linear map $\mathcal{H}_{N-1} \to \mathcal{H}_N$, by the action on a basis $\Upsilon_j^\alpha v_{\mathbf{w}} := v_{\Upsilon_j^\alpha \mathbf{w}}$. Using such maps we can characterize completely the specializations of $\Psi_N(\mathbf{w})$ in terms of solutions of the exchange equations of a smaller system[3]

***Theorem* 13** (Recursion)**.** *Upon specializations $z_j = \tau_1$ or $z_j = \nu_N$ we have the following identities*

$$\Psi_N(\mathbf{z})|_{z_j=\tau_1} = \kappa_N^b(\mathbf{z}_{\widehat{j}})\ \Upsilon_j^1 \tilde{\Psi}_{N-1}(\mathbf{z}_{\widehat{j}}) \tag{40}$$

$$\Psi_N(\mathbf{z})|_{z_j=\nu_N} = \kappa_N^t(\mathbf{z}_{\widehat{j}})\ \Upsilon_j^N \Psi_{N-1}(\mathbf{z}_{\widehat{j}}), \tag{41}$$

where $\tilde{\Psi}_{N-1}(\mathbf{z})$ is obtained from $\Psi_{N-1}(\mathbf{z})$ by renaming $\tau_i, \nu_i \to \tau_{i+1}, \nu_{i+1}$, and

$$\kappa_N^b(\mathbf{z}) = \prod_{\alpha=2}^{N} (\tau_1 - \nu_\alpha)^{\alpha-1} \prod_{i=1}^{N-1} (z_i - \tau_1) \tag{42}$$

$$\kappa_N^t(\mathbf{z}) = \prod_{\alpha=1}^{N-1} (\tau_\alpha - \nu_N)^{N-\alpha} \prod_{i=1}^{L-1} (z_i - \nu_N). \tag{43}$$

[3]Here and in the following, for a finite or infinite string of ordered variables like $\mathbf{z} = \{z_1, z_2, \dots\}$, and a set if integers I, the notation $\mathbf{z}_{\widehat{I}}$ means

$$\mathbf{z}_{\widehat{I}} = \mathbf{z} \setminus \{z_i | i \in I\},$$

keeping the order inherited from $\mathbf{z}$.

Proof. The proof of the two eqs.(40,41) is completely similar, hence we prove only eq.(41). As we already know, Corollary 9 tells us that $\Psi_N(\mathbf{z})|_{z_j=\nu_N}$ is in the image of Υ_j^N, then call $\bar{\Psi}^{(j)}_{N-1}(\mathbf{z}_{\hat{j}})$ its unique preimage. Actually, thanks to Proposition 12, we have that

$$\bar{\Psi}^{(j)}_{N-1}(\mathbf{z}_{1,\dots,L-1}) = \bar{\Psi}^{(k)}_{N-1}(\mathbf{z}_{1,\dots,L-1})$$

for $j \neq k$, hence we can suppress the upper label (j). Since Υ_j^N intertwines the $\check{R}$ matrices acting on sites different from $j-1$ and j

$$\begin{aligned} \check{R}_i(z,w)\Upsilon_j^N &= \Upsilon_j^N \check{R}_i(z,w) & \qquad & i < j-1 \\ \check{R}_{i+1}(z,w)\Upsilon_j^N &= \Upsilon_j^N \check{R}_i(z,w) & \qquad & i > j-1 \end{aligned}$$

it is obvious that $\bar{\Psi}_{N-1}(\mathbf{z}_{1,\dots,L-1})$ satisfies all the exchange equations (23) for $i \neq j-1$. But since $\bar{\Psi}_{N-1}(\mathbf{z}_{1,\dots,L-1})$ doesn't depend on j it must satisfy also the exchange equation for $i = j-1$. Hence by unicity of the solution of the exchange equations we have

$$\bar{\Psi}_{N-1}(\mathbf{z}_{1,\dots,L-1}) \propto \Psi_{N-1}(\mathbf{z}_{1,\dots,L-1}).$$

The proportionality factor is fixed by looking at the specialization of the component associated to the configuration $12\dots N$. $\square$

While the recursion relations of Theorem 13 do not allow to reconstruct recursively the full vector $\Psi_N(\mathbf{z})$ (recall that $\deg_{z_j} \Psi_N(\mathbf{z}) = N-1$), it can be used to determine the form of certain families of components. For $1 \leq \beta \leq N$ define

$$\mathbf{w}^{(\beta,N)} = 12\dots\widehat{\beta}\dots N\beta$$

Thanks to Propositions 8 and 10 we know that the component $\psi_{\mathbf{w}^{(\beta,N)}}(\mathbf{z})$ has the following form

$$\psi_{\mathbf{w}^{(\beta,N)}}(\mathbf{z}) = \phi_N^{(\beta)}(\mathbf{z})\widetilde{\psi}_{\mathbf{w}^{(\beta,N)}}(\mathbf{z}_{\widehat{N}}) \tag{44}$$

where $\phi_N^{(\beta)}(\mathbf{z}) = G_{[1,\dots,N];\mathbf{w}^{(\beta,N)};N,N}(\mathbf{z})G_{[1\dots\widehat{\beta}\dots N];\mathbf{w}^{(\beta,N)};1,N-1}(\mathbf{z})$ and $\widetilde{\psi}_{\mathbf{w}^{(\beta,N)}}(\mathbf{z}_{\widehat{N}})$ is a symmetric polynomial of degree 1 in each of its variables. In this case the recursion relations (40,41), are enough to completely fix $\widetilde{\psi}_{\mathbf{w}^{(\beta,N)}}(\mathbf{z}_{\widehat{N}})$ in a recursive way. For this we introduce the following family of polynomials in the variables $\mathbf{z}, \mathbf{t}, \mathbf{v}$, indexed by two non negative integers r and s

$$\mathfrak{S}^{r,s}(\mathbf{z},\mathbf{t},\mathbf{v}) = \prod_{\substack{1\leq\alpha\leq r\\ 1\leq\beta\leq s}}(t_\alpha - v_\beta)\oint_{\mathbf{t}}\frac{dw}{2\pi i}\frac{\prod_{j=1}^{r+s-2}(z_i - w)}{\prod_{\alpha=1}^{r}(w-t_\alpha)\prod_{\beta=1}^{s}(w-v_\beta)}, \tag{45}$$

where the contour integration encircles only the poles at $\mathbf{t}$. It is easy to see that the polynomials $\mathfrak{S}^{r,s}(\mathbf{z},\mathbf{t},\mathbf{v})$ are fully characterized (by Lagrange interpolation) by the following recursion relations

$$\mathfrak{S}^{r,s}(\mathbf{z},\mathbf{t},\mathbf{v})|_{z_j=t_\alpha} = -\prod_{1\le\beta\le s}(t_\alpha - v_\beta)\mathfrak{S}^{r-1,s}(\mathbf{z}_{\widehat{j}},\mathbf{t}_{\widehat{\alpha}},\mathbf{v}), \tag{46}$$

$$\mathfrak{S}^{r,s}(\mathbf{z},\mathbf{t},\mathbf{v})|_{z_j=v_\beta} = -\prod_{1\le\alpha\le r}(t_\alpha - v_\beta)\mathfrak{S}^{r,s-1}(\mathbf{z}_{\widehat{j}},\mathbf{t},\mathbf{v}_{\widehat{\beta}}), \tag{47}$$

for $1\le\alpha\le r, 1\le\beta\le s$, $1\le j\le r+s-2$ and boundary conditions

$$\mathfrak{S}^{0,s}(\mathbf{z},\mathbf{t},\mathbf{v}) = \mathfrak{S}^{r,0}(\mathbf{z},\mathbf{t},\mathbf{v}) = 0.$$

***Proposition* 14.**

$$\widetilde{\psi}_{\mathbf{w}(\beta,N)}(\mathbf{z}) = \prod_{\substack{1\le\alpha<\gamma\le\beta\\ \& \\ \beta\le\alpha<\gamma\le N}}(\tau_\alpha-\nu_\gamma)^{\gamma-\alpha-1}\prod_{1\le\alpha<\beta<\gamma\le N}(\tau_\alpha-\nu_\gamma)^{\gamma-\alpha-2} \\ \mathfrak{S}^{\beta,N-\beta+1}(\mathbf{z},\{\tau_1,\tau_2\ldots,\tau_\beta\},\{\nu_\beta,\nu_{\beta+1},\ldots,\nu_N\}) \tag{48}$$

Proof. As mentioned before, being $\widetilde{\psi}_{\mathbf{w}(\beta,N)}(\mathbf{z})$ a symmetric polynomial of degree 1 in each variable, it is completely determined by the recursion relations (40,41). Therefore it is sufficient to check that plugging eq.(48) into eq.(44), we obtain a family of polynomials that satisfy the recursion relations. This is readily done using the recursions for $\mathfrak{S}^{r,s}(\mathbf{z},\mathbf{t},\mathbf{v})$, eqs.(46,47) □

At this point we remark the appearance of double Schubert polynomials. Indeed, let $\sigma(h,N)\in\mathcal{S}^N$ be the permutation defined by

$$\sigma(\beta,N) = (1,\beta+1,\beta+2,\ldots,N,2,3,\ldots,\beta).$$

as will be shown in Appendix A, the polynomial $\mathfrak{S}^{\beta,N-\beta+1}(\infty,\mathbf{t},\mathbf{v})$ is the double Schubert polynomial in the variables $\mathbf{t},\mathbf{v}$ associated to the permutation $\sigma(N-\beta+1,N)$

$$\mathfrak{S}^{\beta,N-\beta+1}(\infty,\mathbf{t},\mathbf{v}) = \mathfrak{S}_{\sigma(N-\beta+1,N)}(\mathbf{t},\mathbf{v}). \tag{49}$$

5.4. **Descending configurations.** Let $\mathbf{w}(N,h)$ be the configuration given by

$$\mathbf{w}(N,h) = h(h-1)\ldots 21(h+1)(h+2)\ldots N$$

As particular cases we have that

$$\mathbf{w}(N,1) = 12\ldots(N-1)N, \qquad \mathbf{w}(N,N) = N(N-1)\ldots 21$$

One of the key results of this paper is the formula for $\psi_{\mathbf{w}(N,h)}(\mathbf{z})$, which is the content of the following

***Theorem* 15.** *Let $1 \leq h \leq N$, the polynomial $\psi_{\mathbf{w}(N,h)}(\mathbf{z})$ has the following form*

$$\psi_{\mathbf{w}(N,h)}(\mathbf{z}) = \phi^{(N,h)}(\boldsymbol{\tau}, \boldsymbol{\nu}) \\ G_{[h+1,N],\mathbf{w}(N,h),1,L}(\mathbf{z}) \prod_{\beta=1}^{h} \mathfrak{S}^{(\beta,N-\beta+1)}(\mathbf{z}_{\widehat{h-\beta+1}}, \boldsymbol{\tau}, \boldsymbol{\nu}^c), \tag{50}$$

where

$$\phi^{(N,h)} = \prod_{1\leq\alpha\leq h<\gamma\leq N} (\tau_\alpha - \nu_\gamma)^{\gamma-h-1} \prod_{h<\alpha<\gamma\leq N} (\tau_\alpha - \nu_\gamma)^{\gamma-\alpha-1},$$

where for a fixed N, $\boldsymbol{\nu}^c$ correspond to $\boldsymbol{\nu}$ taken in reversed order starting from N, i.e.

$$\nu_i^c = \nu_{N-i+1}. \tag{51}$$

For later reference let us write explicitly the case $h = 1$, which provides a nice factorized expression for $\psi_{N(N-1)\dots 1}(\mathbf{z})$

$$\psi_{N(N-1)\dots 1}(\mathbf{z}) = \prod_{\beta=1}^{N} \mathfrak{S}^{(\beta,N-\beta+1)}(\mathbf{z}_{\widehat{N-\beta+1}}, \boldsymbol{\tau}, \boldsymbol{\nu}^c). \tag{52}$$

Let us also notice that as a corollary of Theorem 15, by setting $\mathbf{z} = \infty$ we prove a generalization of Lam-Williams Conjectures 3 and 4 [25]

***Corollary* 16.** *With the normalization eq.(1) the component reads*

$$\psi_{\mathbf{w}(N,h)} = \phi^{(N,h)}(\boldsymbol{\tau}, \boldsymbol{\nu}) \prod_{\beta}^{h} \mathfrak{S}_{\sigma(\beta,N)}(\boldsymbol{\tau}, -\boldsymbol{\nu}^c).$$

For the proof of Theorem 15 we need some preparatory steps. Let us introduce the families $\mathcal{D}(N,h)$, which consist of configurations containing a sub-string of consecutive entries of the form

$$h(h-1)\dots 1.$$

Notice that we have $\mathbf{w}(N,h) \in \mathcal{D}(N,h)$ and $\mathcal{D}(N,h) \subset \mathcal{D}(N,k)$ for $k \leq h$. Any configuration $\mathbf{w} \in \mathcal{D}(N,h)$ can be obtained from any other configuration $\widetilde{\mathbf{w}} \in \mathcal{D}(N,h)$ by a sequence of permutations $S_1, \dots, S_\ell$

$$\mathbf{w} = S_\ell \dots S_2 S_1 \widetilde{\mathbf{w}} \tag{53}$$

with S_j being either a transposition of consecutive entries $h < w_{i+1} < w_i$, $s_i : w_i w_{i+1} \mapsto w_{i+1} w_i$ or a permutation σ_i moving w_i from the left of the string $h(h-1)\dots 1$ to its right, i.e.

$$\sigma_i : \mathbf{w}_L w_i h(h-1)\dots 1\mathbf{w}_R \mapsto \mathbf{w}_L h(h-1)\dots 1 w_i \mathbf{w}_R.$$

***Proposition* 17.** *Let $\widetilde{\mathbf{w}} \in \mathcal{D}(N,h)$, where the sub-string $h(h-1)\dots 1$ goes from $\tilde{j}+1$ to $\tilde{j}+h$. Suppose $\psi_{\widetilde{\mathbf{w}}}(\mathbf{z})$ to be of the form*

$$\psi_{\widetilde{\mathbf{w}}}(\mathbf{z}) = \psi^{(0)}_{\widetilde{\mathbf{w}}}(\mathbf{z})\bar{\psi}_{N;\tilde{j}+1,\tilde{j}+h}(\mathbf{z})$$

with $\psi^{(0)}_{\widetilde{\mathbf{w}}}(\mathbf{z})$ a polynomial symmetric in the variables $\mathbf{z}_{\widehat{[\tilde{j}+1,\dots,\tilde{j}+h])}}$ and

$$\bar{\psi}_{N;\tilde{j}+1,\tilde{j}+h}(\mathbf{z}) := \prod_{\beta=1}^{h} \mathfrak{S}^{(\beta,N-\beta+1)}(\mathbf{z}_{\widehat{\tilde{j}+\beta}}, \boldsymbol{\tau}, -\boldsymbol{\nu}^c).$$

Then the same factorization holds for the components associated to any $\mathbf{w} \in \mathcal{D}(N,h)$ with sub-string $h(h-1)\dots 1$ between positions $j+1$ and $j+h$, i.e. the component $\psi_{\mathbf{w}}(\mathbf{z})$ is also of the form

$$\psi_{\mathbf{w}}(\mathbf{z}) = \psi^{(0)}_{\mathbf{w}}(\mathbf{z})\bar{\psi}_{N;j+1,j+h}(\mathbf{z}), \tag{54}$$

for some $\psi^{(0)}_{\mathbf{w}}(\mathbf{z})$, symmetric in $\mathbf{z}_{\widehat{[j+1,j+h]}}$.

Proof. Call $d(\widetilde{\mathbf{w}},\mathbf{w})$ the minimal ℓ for which we have a sequence of permutations realizing eq.(53). We prove the statement by induction on $d(\widetilde{\mathbf{w}},\mathbf{w})$. For $d(\widetilde{\mathbf{w}},\mathbf{w})=0$ there is nothing to prove. Now consider $\ell = d(\widetilde{\mathbf{w}},\mathbf{w}) > 0$, this means that we can write $\mathbf{w} = S_\ell \dots S_1 \mathbf{w}$, for some S_i as defined above. By induction we know that for $\mathbf{w}' = S_{\ell-1}\dots S_1\mathbf{w}$, $\psi_{\mathbf{w}'}(\mathbf{z})$ is of the form eq.(54). For S_ℓ there are two possibilities either $S_\ell = s_i$ or $S_\ell = \sigma_i$ for some s_i or σ_i.

If $S_\ell = s_i$, it means that $\psi_{\mathbf{w}}(\mathbf{z}) = \pi_i(w'_i, w'_{i+1})\psi_{\mathbf{w}'}(\mathbf{z})$. The polynomial $\psi_{\mathbf{w}'}(\mathbf{z})$ is of the form eq.(54) with the factor $\bar{\psi}_{\mathbf{m},\rho,\mathbf{w}}(\mathbf{z})$ symmetric in z_i, z_{i+1}, therefore we have $\psi_{\mathbf{w}}(\mathbf{z}) = \left[\pi_i(w'_i, w'_{i+1})\psi^{(0)}_{\mathbf{w}}(\mathbf{z})\right]\bar{\psi}_{N,j'+1,j'+h}(\mathbf{z})$, which is again of the form eq.(54).

If $S_\ell = \sigma_i$ then $\psi_{\mathbf{w}}(\mathbf{z}) = \pi_{\sigma_i}(\mathbf{w}')\psi_{\mathbf{w}'}(\mathbf{z})$. Since $\psi_{\mathbf{w}'}(\mathbf{z})$ is of the form eq.(54), we can use Proposition 24[4], with $u = w_i$,

$$K(z_i; \mathbf{z}_{i+1,\dots,i+h}) = \psi^{(0)}_{\mathbf{w}}(\mathbf{z}), \qquad F_j(w) = \frac{\prod_{\substack{1\le\ell\le N \\ \ell\notin[\widehat{i+1,i+h}]}}(z_\ell - w)}{\prod_{\beta=h}^{N}(w-\nu_\beta)}.$$

Then eq.(66) allows to conclude that $\psi_{\mathbf{w}}(\mathbf{z})$ is of the desired form. □

We can now pass to the proof of Theorem 15.

Proof. of Theorem 15. The proof proceeds by a double induction on N and on h.

For $N=1$ the statement is trivial, while the case $h=1$ holds because in such case $\mathbf{w}(N,1) = 12\dots N$ and eq.(50) coincides with eq.(34).

[4]Up to a translation in the indices of the variables $z_k \to z_{k-i+1}$.

Assuming $N, h > 1$ we proceed by factor exhaustion. The presence of the factor $G_{[h+1,N],\mathbf{w}(N,h),1,L}(\mathbf{z})$ is ensured by Propositions 8 and 10. By induction we know that $\psi_{\mathbf{w}(N,h-1)}(\mathbf{z})$ is of the form eq.(50) and in particular of the form eq.(54), therefore it follows from Proposition 17 that the components corresponding to configurations in $\mathcal{D}(N, h-1)$ have the same form. Since $\mathbf{w}(N,h) \in \mathcal{D}(N, h-1)$, we conclude in particular that $\psi_{\mathbf{w}(N,h)}(\mathbf{z})$ contains the factor

$$\prod_{\beta=1}^{h-1} \mathfrak{S}^{(\beta,N-\beta+1)}(\mathbf{z}_{\widehat{h-\beta+1}}, \boldsymbol{\tau}, \boldsymbol{\nu}^c),$$

which is prime with $G_{[h+1,N],\mathbf{w}(N,h),1,L}(\mathbf{z})$. The remaining factor $g(\mathbf{z})$ is a symmetric polynomial in the variables $z_h, z_{h+1}, \ldots, z_N$, of degree 1 in each of these variables. Therefore, in order to check that

$$g(\mathbf{z}) = \phi^{(N,h)}(\boldsymbol{\tau}, \boldsymbol{\nu})\mathfrak{S}^{(h,N-h+1)}(\mathbf{z}_{\widehat{1}}, \boldsymbol{\tau}, \boldsymbol{\nu}^C)$$

it is enough to check that eq.(50) holds when specialized at two distinct values of z_i for $i \in [h, N]$. For $i = N$, using the recursions of Theorem 13, the specialization $\psi_{\mathbf{w}(N,h)}(\mathbf{z})|_{z_N=\nu_N}$ can be written in terms of $\psi_{\mathbf{w}(N-1,h)}(\mathbf{z}_{\widehat{N}})$, which by induction (it corresponds to the case $N-1$) is given by the expression of eq.(50). It is not difficult to check that using eq.(50) for $\psi_{\mathbf{w}(N,h)}(\mathbf{z})$ and specializing $z_N = \nu_N$ one obtains the same result. In the same way one can check the specialization $z_h = \tau_1$. □

Another result that can be obtained by using Theorem 15 concerns the primality of the components of $\Psi_N(\mathbf{z})$

***Theorem* 18.** *With the normalization given by eq.(34) the components of $\Psi_N(\mathbf{z})$, as functions of $\boldsymbol{\tau}, \boldsymbol{\nu}$ and $\mathbf{z}$, are prime polynomials with integer coefficients.*

Proof. First we notice that if $F(\boldsymbol{\tau}, \boldsymbol{\nu}; \mathbf{z})$ is a polynomial in $\boldsymbol{\tau}, \boldsymbol{\nu}, \mathbf{z}$ with integer coefficients and $\pi_i(\beta, \alpha)F(\boldsymbol{\tau}, \boldsymbol{\nu}; \mathbf{z})$ is also polynomial in $\boldsymbol{\tau}, \boldsymbol{\nu}, \mathbf{z}$, then it must have integer coefficients as well. Therefore, since all the components can be obtained from $\psi_{12\ldots N}(\mathbf{z})$ (which has integer coefficients) by action of operators $\pi_i(\beta, \alpha)$, once we will have proven that all components are polynomial in $\boldsymbol{\tau}, \boldsymbol{\nu}, \mathbf{z}$ we shall automatically get that their components are integer.

From their formulas, we see that $\psi_{12\ldots N}(\mathbf{z})$ and $\psi_{N(N-1)\ldots 1}(\mathbf{z})$ are prime polynomials in all the variables $\mathbf{z}, \boldsymbol{\tau}, \boldsymbol{\nu}$ (they have no common polynomial factor), therefore the only thing that remains to be proven is the polynomiality in $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ of all the other components. Any $\psi_{\mathbf{w}}(\mathbf{z})$ can be obtained from $\psi_{N(N-1)\ldots 1}(\mathbf{z})$, by sequential action of the operators $\pi_\ell(\alpha, \beta)$ with $1 \le \ell \le L-1$. In particular if for $i < j$,

$w_i < w_j$, then exactly one of the members of the sequence of operators $\pi_\ell(\alpha,\beta)$ is of the form $\pi_\ell(w_i,w_j)$ for some ℓ. Therefore, in principle we could get a factor at denominator of the form $\tau_{w_i} - \nu_{w_j}$, screwing up the polynomiality in $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$. We have to make sure this does not happen. In facts the reasoning above can be reversed giving us some positive information, namely it tells us that if for $j < i$, $w_i < w_j$, then at denominator we do not have a factor of the form $\tau_{w_i} - \nu_{w_j}$, because there is no way this could have been arisen. Now let us come back to the case $i < j$, $w_i < w_j$: upon rotating by $h = L+1-j$ steps we get a new configuration $\tilde{\mathbf{w}} = R^h\mathbf{w}$ with $\tilde{w}_1 = w_j$ and $\tilde{w}_{L+1+i-j} = w_i$. Since $L+1+i-j > 1$ we conclude that $\psi_{R^h\mathbf{w}}(\mathbf{z})$ (and henceforth $\psi_{\mathbf{w}}(\mathbf{z})$) does not have the factor $\tau_{w_i} - \nu_{w_j}$ at denominator. □

A corollary of the previous result is Theorem 1.

5.5. **Factorization and normalization.** Let us now draw a few consequences of Theorem 15.

***Theorem* 19.** *Let $1 \le h \le k \le N$ and $\mathbf{w}$ of the form*

$$\mathbf{w} = \mathbf{w}^{(L)}k(k-1)\dots(h+1)h\mathbf{w}^{(R)}$$

with $w_i^{(L)} > k$ and $w_i^{(r)} < h$, then $\psi_{\mathbf{w}^{(L)}k(k-1)\dots(h+1)h\mathbf{w}^{(R)}}(\mathbf{z})$ has a factor of the form

$$\prod_{\beta=h}^{k} \mathfrak{S}^{(\beta,N-\beta+1)}(\mathbf{z}_{\widehat{N-\beta+1}}, \boldsymbol{\tau}, \boldsymbol{\nu}^c) \tag{55}$$

Proof. The configuration $\mathbf{w} = \mathbf{w}^{(L)}k(k-1)\dots(h+1)h\mathbf{w}^{(R)}$ is obtained from the descending configurations $N(N-1)\dots1$ through transpositions involving only the first $N-k$ and the last $h-1$ positions. Therefore $\psi_{\mathbf{w}^{(L)}k(k-1)\dots(h+1)h\mathbf{w}^{(R)}}(\mathbf{z})$ is obtained from $\psi_{N(N-1)\dots1}(\mathbf{z})$ by the action of operators $\pi_i(\alpha,\beta)$ with $1 \le i \le N-k-1$ or $N-h+1 \le i \le N-1$. Any factor of $\psi_{N(N-1)\dots1}(\mathbf{z})$ which is symmetric in the first $N-k$ and the last $h-1$ variables is preserved by the action of such operators and is a factor of $\psi_{\mathbf{w}^{(L)}k(k-1)\dots(h+1)h\mathbf{w}^{(R)}}(\mathbf{z})$. The product $\prod_{\beta=h}^{k} \mathfrak{S}^{(\beta,N-\beta+1)}(\mathbf{z}_{\widehat{N-\beta+1}}, \boldsymbol{\tau}, \boldsymbol{\nu}^c)$ is such a factor. □

In the same spirit we have the following factorization result

***Theorem* 20.** *Suppose that $\mathbf{w} = \mathbf{x}\mathbf{y}$ with $x_i > y_j$, with $\ell(\mathbf{x}) = k$, then*

$$\psi_{\mathbf{x}\mathbf{y}}(\mathbf{z}) = \psi^{(1)}_{\mathbf{x}}(\mathbf{z})\psi^{(2)}_{\mathbf{y}}(\mathbf{z}), \tag{56}$$

where $\psi^{(1)}_{\mathbf{x}}(\mathbf{z})$ is symmetric in $z_{k+1},\dots,z_N$ and depends only on $\mathbf{x}$, while $\psi^{(2)}_{\mathbf{y}}(\mathbf{z})$ is symmetric in $z_1,\dots,z_k$ and depends only on $\mathbf{y}$.

Proof. Using eq.(52), we write $\psi_{N(N-1)\dots 21}(\mathbf{z})$ as

$$\psi_{N(N-1)\dots 21}(\mathbf{z}) = \psi^{(1)}_{N,k}(\mathbf{z})\psi^{(2)}_{N,k}(\mathbf{z})$$

with

$$\psi^{(1)}_{N,k}(\mathbf{z}) = \prod_{\beta=N-k+1}^{N} \mathfrak{S}^{(\beta,N-\beta+1)}(\mathbf{z}_{\widehat{N-\beta+1}}, \boldsymbol{\tau}, -\boldsymbol{\nu}^c)$$

$$\psi^{(2)}_{N,k}(\mathbf{z}) = \prod_{\beta=1}^{N-k} \mathfrak{S}^{(\beta,N-\beta+1)}(\mathbf{z}_{\widehat{N-\beta+1}}, \boldsymbol{\tau}, -\boldsymbol{\nu}^c).$$

Notice that $\psi^{(1)}_{N,k}(\mathbf{z})$ is symmetric in the last $N-k$ variables, while $\psi^{(2)}_{N,k}(\mathbf{z})$ is symmetric in the first k variables.

Any configurations $\mathbf{w} = \mathbf{xy}$ with $x_i > y_j$, with $\ell(\mathbf{x}) = k$ is obtained from $N(N-1)\dots 1$ by separate transpositions of the first k and last $N-k$ and therefore $\psi_{\mathbf{xy}}(\mathbf{z})$ is obtained from $\psi_{N(N-1)\dots 21}(\mathbf{z})$ by action of operators $\pi_i(\alpha,\beta)$ with $i \neq k$. Being $\psi^{(2)}_{N,k}(\mathbf{z})$ symmetric in the first k variables, the operators $\pi_i(\alpha,\beta)$ with $i < k$ act only on the first factor $\psi^{(1)}_{N,k}(\mathbf{z})$, viceversa the operators $\pi_i(\alpha,\beta)$ with $i > k$ act only on the second factor $\psi^{(2)}_{N,k}(\mathbf{z})$ and therefore we get the factorization of eq.(52). $\square$

Theorem 20 means in particular that for the stationary measure, under the conditioning that the configuration $\mathbf{w}$ splits as $\mathbf{w} = \mathbf{xy}$, with $x_i > y_j$, then $\mathbf{x}$ and $\mathbf{y}$ are independent. This fact has already been proven for the case $\mathbf{z} = \infty$ and $\nu_\alpha = 0$ in [2] using the multiline queues representation of $\psi_{\mathbf{w}}$.

The next result concerns the *partition function* $\mathcal{Z}_N(\mathbf{z}, \boldsymbol{\tau}, \boldsymbol{\nu})$, that is the sum of all the components

$$\mathcal{Z}_N(\mathbf{z}, \boldsymbol{\tau}, \boldsymbol{\nu}) := \sum_{\ell(\mathbf{w})=N} \psi_{\mathbf{w}}(\mathbf{z}). \tag{57}$$

In order to write the formula for $\mathcal{Z}_N(\mathbf{z}, \boldsymbol{\tau}, \boldsymbol{\nu})$ we need a little bit of further notation. For an ordered set of N variables $\mathbf{z} = \{z_1, \dots, z_N\}$ and a permutation $\sigma \in \mathcal{S}_N$ we write $\mathbf{z}_\sigma := \{z_{\sigma(1)}, \dots, z_{\sigma(N)}\}$. For any function $f(\mathbf{z})$ of N variables, we call $\mathrm{Sym}\,[f(\mathbf{z})]$ its symmetrized version, i.e.

$$\mathrm{Sym}\,[f(\mathbf{z})] = \sum_{\sigma \in \mathcal{S}_N} f(\mathbf{z}_\sigma).$$

We have the following

***Theorem* 21** (Partition function)**.**

$$\mathcal{Z}_N(\mathbf{z},\boldsymbol{\tau},\boldsymbol{\nu}) = \frac{1}{\prod_{1\le\alpha<\beta\le N}(\tau_\alpha-\nu_\beta)^{\beta-\alpha}} Sym\left[\frac{\psi_{1\cdots(N-1)N}(\mathbf{z}_{w_0})\psi_{N(N-1)\cdots 1}(\mathbf{z})}{\Delta(\mathbf{z})}\right]$$

Where w_0 *is the longest permutation in* $\mathcal{S}_N$ $w_0 = (N, N-1, \dots, 2, 1)$.

Proof. As we noticed before, any component can be obtained from $\psi_{N(N-1)\cdots 1}(\mathbf{z})$ by sequential action of operators $\pi_i(\alpha,\beta)$, with $1\le i\le N-1$. By expanding the divided difference operator we get

$$\psi_{\mathbf{w}}(\mathbf{z}) = \sum_{\sigma\in\mathcal{S}_N} k_{\mathbf{w},\sigma}(\mathbf{z})\ \psi_{N(N-1)\cdots 1}(\mathbf{z}_\sigma) \tag{58}$$

for some coefficients $k_{\mathbf{w},\sigma}(\mathbf{z})$. It is not difficult to see that if $\ell(\sigma)$ is larger than the length of the shortest permutation mapping $N(N-1)\cdots 1$ to $\mathbf{w}$ then $k_{\mathbf{w},\sigma}(\mathbf{z}) = 0$. In particular if σ is the longest permutation w_0 in $\mathcal{S}_N$ then $k_{\mathbf{w},w_0}(\mathbf{z}) \neq 0$ only for $\mathbf{w} = 12\cdots N$. In such a case, by explicitly expanding the divided difference operators, we easily find

$$k_{12\cdots N,w_0}(\mathbf{z}) = \frac{(-1)^{\frac{N(N-1)}{2}}\psi_{1\cdots(N-1)N}(\mathbf{z})}{\left(\prod_{1\le\alpha<\beta\le N}(\tau_\alpha-\nu_\beta)^{\beta-\alpha}\right)\Delta(\mathbf{z})}.$$

By using eq.(58) we can write $\mathcal{Z}(\mathbf{z},\boldsymbol{\tau},\boldsymbol{\nu})$ as

$$\mathcal{Z}(\mathbf{z},\boldsymbol{\tau},\boldsymbol{\nu}) = \sum_{\sigma\in\mathcal{S}_N} K_\sigma(\mathbf{z})\ \psi_{N(N-1)\cdots 1}(\mathbf{z}_\sigma).$$

where $K_\sigma(\mathbf{z}) = \sum_{\mathbf{w}} k_{\mathbf{w},\sigma}(\mathbf{z})$. In the case $\sigma = w_0$, thanks to the discussion above, the sum giving $K_{w_0}(\mathbf{z})$ reduces to a single term

$$K_{w_0}(\mathbf{z}) = \sum_{\mathbf{w}} k_{\mathbf{w},w_0}(\mathbf{z}) = k_{12\cdots N,w_0}(\mathbf{z}).$$

On the other hand we know that $\mathcal{Z}(\mathbf{z},\boldsymbol{\tau},\boldsymbol{\nu})$ is symmetric in $\mathbf{z}$, therefore we have

$$K_\sigma(\mathbf{z}) = K_e(\mathbf{z}_\sigma),$$

which concludes the proof. □

Since both $\psi_{1\cdots(N-1)N}(\mathbf{z}_{w_0})$ and $\psi_{N(N-1)\cdots 1}(\mathbf{z})$ have factorized expression, the formula obtained in Theorem 21 can be recast in a determinantal form

$$\mathcal{Z}(\mathbf{z},\boldsymbol{\tau},\boldsymbol{\nu}) = \frac{1}{\Delta(\mathbf{z})\prod_{1\le\alpha<\beta\le N}(\tau_\alpha-\nu_\beta)}\ \det_{1\le\alpha,\beta\le N} M^{(N)}_{\alpha,\beta} \tag{59}$$

where

$$M^{(N)}_{\alpha,\beta} = \mathfrak{S}^{(\beta,N-\beta+1)}(\mathbf{z}_{\widehat{\alpha}},\boldsymbol{\tau},\boldsymbol{\nu}^c)\prod_{1\le\gamma<N-\beta+1}(z_\alpha-\tau_\gamma)\prod_{N-\beta+1<\gamma\le N}(z_\alpha-\nu_\gamma).$$

Appendix A. Schubert polynomials

The Double Schubert polynomials play an important role in the geometry of flag varieties, where they represent equivariant cohomology classes, and in the combinatorics of the Bruhat order of the symmetric group (see [28, 29, 30]). In this Appendix we recall their definition and provide an explicit formula for a certain class of permutations.

Let $\mathbf{t} = \{t_1, t_2, \dots\}$ and $\mathbf{v} = \{v_1, v_2, \dots\}$ be two infinite sets of variables. In this section we use the divided difference operators in the variables $\mathbf{t}$, so here s_i is the transposition of the variables $t_i \leftrightarrow t_{i+1}$

$$\partial_i = \frac{1 - s_i}{t_i - t_{i+1}}.$$

These operators satisfy the following relations

$$\begin{aligned} \partial_i^2 &= 0 \\ \partial_i \partial_j &= \partial_j \partial_i \qquad |i-j| > 1 \\ \partial_i \partial_{i+1} \partial_i &= \partial_{i+1} \partial_i \partial_{i+1} \end{aligned} \tag{60}$$

Let $\mathcal{S}^\infty$ be the infinite symmetric group, the algebra generated by ∂_i for $i \geq 1$ has a basis indexed by permutations $\sigma \in \mathcal{S}^\infty$. Let $s_{i_\ell} \cdots s_{i_1}$ be a reduced decomposition of σ, then

$$\partial_\sigma := \partial_{i_\ell} \cdots \partial_{i_1}$$

is well defined, i.e. it is the same for different reduced decomposition of the same permutation.

Definition (Double Schubert polynomials)**.** *The double Schubert polynomials are a family of polynomials $\mathfrak{S}_\sigma(\mathbf{t}, \mathbf{v})$ in the variables $\mathbf{t}, \mathbf{v}$ indexed by permutations $\sigma \in \mathcal{S}^\infty$. For $\sigma \in \mathcal{S}^N \subset \mathcal{S}^\infty$, $\mathfrak{S}_\sigma(\mathbf{t}, \mathbf{v})$ is defined by*

$$\mathfrak{S}_\sigma(\mathbf{t}, \mathbf{v}) = \partial_{\sigma^{-1} w_N} \prod_{i+j \leq N} (t_i - v_j) \tag{61}$$

where w_N is the longest permutation in $\mathcal{S}^N \subset \mathcal{S}^\infty$.

Now let $\sigma(h, N) \in \mathcal{S}^N$ defined by

$$\sigma(h, N) = (1, h+1, h+2, \dots N, 2, 3, \dots, h).$$

We show that

$$\mathfrak{S}_{\sigma(h,N)}(\mathbf{t}, \mathbf{v}) = \oint_{\mathbf{t}} \frac{dw}{2\pi i} \frac{\prod_{\substack{1 \leq \alpha \leq N-h+1 \\ 1 \leq \beta \leq h}} (t_\alpha - v_\beta)}{\prod_{1 \leq \alpha \leq N-h+1} (w - t_\alpha) \prod_{1 \leq \beta \leq h} (w - v_\beta)} \tag{62}$$

First notice that for $\tilde{\sigma}(h,N) = (h+1, h+2, \dots N, 1, 2, 3, \dots, h)$ we have

$$\mathfrak{S}_{\sigma'(h,N)}(\mathbf{t},\mathbf{v}) = \prod_{\substack{1\le\alpha\le N-h\\ 1\le\beta\le h}} (t_\alpha - v_\beta).$$

Then, using the definition eq.(61), we can write

$$\mathfrak{S}_{\sigma(h,N)}(\mathbf{t},\mathbf{v}) = \partial_1 \cdots \partial_{N-h-1}\partial_{N-h}\mathfrak{S}_{\sigma'(h,N)}(\mathbf{t},\mathbf{v})$$

and eq.(62) follows from this general Lemma

***Lemma* 22.** *Let $K(t_1, t_2, \dots, t_k; t_{k+1})$ be a symmetric function in the variables $\{t_1, t_2, \dots, t_k\}$, then the following identity holds*

$$\partial_1 \cdots \partial_{k-1}\partial_k K(t_1, t_2, \dots, t_k; t_{k+1}) = \sum_{i=1}^{k+1} \frac{K(t_1, \dots, \widehat{t_i}, \dots, t_{k+1}; t_i)}{\prod_{1\le j\ne i\le k+1}(t_i - t_j)} \tag{63}$$

Proof. If we act with ∂_j with $1 \le j < k$, on the l.h.s. and use the braiding relations of the operators ∂_i we get zero, therefore the l.h.s. is symmetric in the variables $t_1, \dots, t_k, t_{k+1}$. On the other hand, by developing the action of the divided difference operators, we know that the l.h.s. can be written as

$$\sum_{i=1}^{k+1} K(t_1, \dots, \widehat{t_i}, \dots, t_{k+1}; t_i) G_i(\mathbf{t}).$$

Therefore it is enough to compute one of the coefficients $G_i(\mathbf{t})$. The term $G_1(\mathbf{x})K(t_2, \dots, t_k, t_{k+1}; t_1)$ in the previous equation can be obtained in a unique way from the expansion of the divided difference operators, namely it is given by

$$\frac{1}{x_1 - x_2} s_1 \cdots \frac{1}{t_{k-1} - t_k} s_{k-1} \frac{1}{t_k - t_{k+1}} s_k K(t_1, \dots, t_k; t_{k+1})$$

and hence we have $G_1(\mathbf{t}) = \frac{1}{\prod_{j=2}^{k+1}(t_1 - t_j)}$. □

Appendix B. Technical results

In this Appendix we are going to present some technical results needed for the proof of Theorem 15. Let $\mathbf{z} = \{z_0, z_1, \dots, z_n\}$ and $m \le n$, then define the following functions

$$G^{(F,m)}(\mathbf{z}) := \oint_{\mathbf{t}} \frac{dw}{2\pi i} \frac{\prod_{j=1}^{n}(z_j - w)F(w)}{\prod_{j=1}^{m}(w - \tau_j)} \tag{64}$$

***Proposition* 23.** *The following identity holds*

$$\sum_{j=0}^{n} \frac{G^{(F,m)}(\mathbf{z}_{\widehat{j}}) \prod_{\alpha=1}^{m}(z_j-\tau_\alpha)}{\prod_{0\le i\neq j\le n}(z_j-z_i)} = 0. \tag{65}$$

Proof. Consider the function $\tilde{G}(y) := G^{\left(\frac{F}{y-w},m\right)}(\mathbf{z})\prod_{\alpha=1}^{m}(y-\tau_\alpha)$. It is simple to see that $\tilde{G}(y)$ is polynomial in y of degree strictly less than m. Therefore the contour integral

$$\sum_{j=0}^{n}\frac{\tilde{G}(z_j)}{\prod_{0\le i\neq j\le n}(z_j-z_i)} = \oint_{\mathbf{z}}\frac{dy}{2\pi i}\frac{\tilde{G}(y)}{\prod_{j=0}^{n}(y-z_j)} = 0.$$

Then in order to conclude it is sufficient to notice that

$$G^{\left(\frac{F}{y-w},m\right)}(\mathbf{z})|_{y=z_j} = G^{(F,m)}(\mathbf{z}_{\widehat{j}}).$$

$\square$

***Proposition* 24.** *Let $\mathbf{z}=\{z_1,\dots,z_{h+1}\}$, take $K(z_1;\mathbf{z}_{\widehat{1}})$ to be a symmetric function in the variables $\mathbf{z}_{\widehat{1}}$ then the following identity holds*

$$\begin{aligned}&\pi_h(u;1)\pi_{h-1}(u;2)\cdots\pi_1(u;h)\left(K(z_1;\mathbf{z}_{\widehat{1}})\prod_{j=1}^{h}G^{(F_j,j)}(\mathbf{z}_{\widehat{h-j+2}})\right)=\\ &\sum_{j=1}^{h+1}\frac{K(z_j;\mathbf{z}_{\widehat{j}})\prod_{i=1}^{h}(z_j-\tau_i)}{\prod_{1\le i\neq j\le h+1}(z_j-z_i)}\prod_{j=1}^{h}\left(\frac{(z_j-\nu_u)}{(\tau_j-\nu_u)}G^{(F_j,j)}(\mathbf{z}_{\widehat{h-j+1}})\right).\end{aligned} \tag{66}$$

Proof. We prove the statement by induction on h. For $h=1$ the statement is immediate to check[5].

Now assume $h>1$. We start by applying $\pi_1(u,h)$ on $H(\mathbf{z})$. Since the product of the first $h-1$ terms, $\prod_{j=1}^{h-1}G^{(F_j,j)}(\mathbf{z}_{\widehat{h-j+2}})$, is symmetric in the variables z_1,z_2 it remains as a factors, hence it is sufficient to look at

$$\tilde{K}(z_1,z_2;\mathbf{z}_{\widehat{1,2}}) = \pi_1(u,h)\left(K(z_1;\mathbf{z}_{\widehat{1}})G^{(F_h,h)}(\mathbf{z}_{\widehat{2}};\mathbf{n}_h)\right) \tag{67}$$

Since the function $\tilde{K}(z_1,z_2;\mathbf{z}_{\widehat{1,2}})$ is symmetric in the variables $z_3,\dots,z_{h+1}$, when we proceed with the action of the remaining divided difference

[5]Notice that

$$G^{(F,1)}(\mathbf{z}) = F(\tau_1)\prod_j(z_j-\tau_1).$$

operators $\pi_h(u;1)\cdots\pi_2(u;h-1)$, we are in the case $h-1$ and by induction we get

$$\sum_{j=2}^{h+1}\frac{\tilde{K}(z_1,z_j;\mathbf{z}_{\widehat{1,j}})\prod_{i=1}^{h-1}(z_j-\tau_i)}{\prod_{2\leq i\neq j\leq h+1}(z_j-z_i)}\prod_{j=1}^{h-1}\frac{(z_j-\nu_u)}{(\tau_j-\nu_u)}G^{(F_j,j)}(\mathbf{z}_{\widehat{h-j+1}}). \tag{68}$$

It remains to compute the sum in the previous equation. For this we split $\tilde{K}(z_1,z_j;\mathbf{z}_{\widehat{1,j}})$ in two parts

$$\frac{(z_j-\tau_h)(z_1-\nu_u)}{(\tau_h-\nu_u)(z_1-z_j)}(K(z_1;\mathbf{z}_{\hat{1}})G^{(F_h,h)}(\mathbf{z}_{\hat{j}})-K(z_j;\mathbf{z}_{\hat{j}})G^{(F_h,h)}(\mathbf{z}_{\hat{1}})).$$

Once substituted into eq.(68) the leftmost term provides the terms for $2\leq j\leq h+1$ in the sum in eq.(66). For the first term we are led to consider the sum

$$-\frac{(z_1-\nu_u)K(z_1;\mathbf{z}_{\hat{1}})}{(\tau_h-\nu_u)}\sum_{j=2}^{h+1}\frac{G^{(F_h,h)}(\mathbf{z}_{\hat{j}})\prod_{i=1}^{h}(z_j-\tau_i)}{\prod_{1\leq i\neq j\leq h+1}(z_j-z_i)},$$

which can be easily evaluated using Proposition 23, giving the remaining term in the sum in eq.(66), namely

$$\frac{(z_1-\nu_u)}{(\tau_h-\nu_u)}\frac{K(z_1;\mathbf{z}_{\hat{1}})\prod_{i=1}^{h}(z_1-\tau_i)}{\prod_{2\leq i\leq h+1}(z_1-z_i)}.$$

□

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